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# On the existence of three-dimensional convection in a rectangular box containing fluid-saturated porous material

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On the basis of a stability analysis of finite amplitude, two-dimensional convection, we have determined the dimensions of boxes containing fluid-saturated porous material in which convection is necessarily unsteady or steady and three-dimensional. For certain box sizes, convective rolls are unstable at Rayleigh numbers $Ra$ lower than 380, the value below which rolls are stable forms of convection between infinite parallel planes. For $Ra = 100$ and 200, it appears unlikely that there are any box dimensions for which there is not a stable (possibly multicellular) two-dimensional steady motion. At $Ra = 340$ and 400, boxes in which rolls are unstable have heights which range from one to five times their horizontal dimensions.

## 1. Introduction

Under what conditions will steady, finite amplitude thermal convection be fully three-dimensional? We attempt to answer this question for convection in a rectangular box containing fluid-saturated porous material heated uniformly from below. The existence of three-dimensional convection for certain box sizes over a limited range of Rayleigh numbers has been demonstrated (Holst & Aziz 1972). Such solutions are of fundamental importance for our understanding of the general phenomenon of convection.

Straus (1974) studied the stability of two-dimensional, finite amplitude convective rolls in an infinite slab of porous material saturated with fluid and heated uniformly from below. For Rayleigh numbers greater than $4\pi^2$, the critical value for the onset of convection in an infinite slab (Horton & Rogers 1945; Lapwood 1948), and less than about 380, two-dimensional rolls are stable to infinitesimal roll perturbations with axes perpendicular to the roll axes of the basic state over only a limited range of basic-state horizontal wavenumbers. For Rayleigh numbers larger than about 380, two-dimensional rolls are always unstable. Here the Rayleigh number is $Ra \equiv \alpha gKd\Delta T/\nu k$, where $\alpha$ is the thermal expansivity of the fluid, $g$ is the acceleration due to gravity, $K$ is the permeability of the porous material, $d$ is the thickness of the layer, $\Delta T$ is the positive temperature difference across the layer, $\nu$ is the kinematic viscosity of the fluid and $k$ is the combined thermal diffusivity of the porous medium and fluid.

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Thus, in a horizontally infinite layer we can only say that, for $380 \gtrsim Ra > 4\pi^2$, steady convection is either two-dimensional with limited values of the horizontal wavenumber or three-dimensional; it is not possible to conclude that steady convection will necessarily be three-dimensional. In fact, two-dimensional convection is probably preferred when its horizontal wavenumber lies within the allowed range; however, initial conditions may affect the horizontal planform of the motion. For $Ra \gtrsim 380$, steady convection in an infinite slab is necessarily three-dimensional. Of course, there are circumstances in which steady convection need not exist and only a two- or three-dimensional oscillatory state may be possible (Horne & O'Sullivan 1974; Caltagirone 1975). This is a contingency that we shall not attempt to deal with in this paper. Rather, we shall always ask, if steady convection exists, is it two- or three-dimensional?

In a region of finite dimensions, in particular a rectangular box, it is possible to make more precise statements about the two- or three-dimensional character of steady convection. This is because two-dimensional convection must exactly fit the dimensions of the box. Thus there are two basic wavenumbers of two-dimensional convection in a box, corresponding to rolls with horizontal axes parallel to one of the two pairs of sides of the box. The feasibility of multicellular convection means that the wavenumbers for two-dimensional rolls could be integral multiples of the basic wavenumbers. Although two-dimensional rolls of a given wavenumber may be unstable in an infinite layer, if these rolls were in a box they could be stable, because the orthogonal rolls to which they are potentially unstable in an infinite layer might not fit within the box.

Straus (1974) did not give the wavenumbers of the orthogonal rolls which destabilize a given basic two-dimensional roll convection pattern. Thus one cannot determine from his results whether two-dimensional convection in a rectangular box will be stable or unstable. The major purpose of this paper is to extend the analysis of Straus (1974) to the finite geometry and delineate, for a given Rayleigh number, those box dimensions for which steady convection will necessarily be three-dimensional. With this knowledge it should be possible to calculate numerically the three-dimensional convection patterns. These three-dimensional solutions might then be used as initial states to obtain other three-dimensional solutions for parameter ($Ra$ and basic horizontal wavenumber) values outside the range in which the results of this paper guarantee three-dimensional convection, if in fact conditions are such that the convection realized depends on initial conditions.

## 2. Description of the method and results for $Ra = 100$

Because the mathematical formulation of our problem is identical to that of Straus (1974), we refrain from repeating details; instead, we describe our approach qualitatively and refer the reader to Straus (1974) for a complete description of the analysis.

Consider a rectangular box containing fluid-saturated porous material with height $d$ and horizontal dimensions $\pi d/\alpha$ in the $x$ direction and $\pi d/\beta$ in the $y$ direction. The upper and lower surfaces of the box are maintained at constant temperatures with the lower surface hotter by $\Delta T$. The side walls of the box are insulating impermeable boundaries; the top and bottom of the box are also impermeable. $Ra$, $\alpha$ and $\beta$ are the only dimensionless parameters which determine the nature of the flow and temperature fields in the box.

For convection of infinitesimal amplitude, a linearized analysis suffices to determine the preferred mode of convection. At the onset of convection, the mode may be either two- or three-dimensional, depending on the box dimensions (Beck 1972). However, the linearized stability analysis for the onset of convection in a rectangular box is of little help in determining whether steady, finite amplitude convection is two- or three-dimensional. Instead, the stability of two-dimensional, steady, finite amplitude convection needs to be examined. If the steady two-dimensional flow is unstable, then steady convection must be three-dimensional. However, if the two-dimensional convection is stable, steady three-dimensional flow cannot be assured; indeed, three-dimensional convection can exist only if there is a non-uniqueness associated with initial conditions.

The approach we adopt to delineate regions of parameter $(Ra, \alpha, \beta)$ space in which steady convection is necessarily three-dimensional is as follows. For a given Rayleigh number, we first establish what modes of steady two-dimensional convection can exist, on the basis of linear stability theory, in a box whose dimensions are characterized by $\alpha$ and $\beta$. Consider two-dimensional rolls aligned along the $y$ axis ('alpha rolls'). The marginal-stability curves are given by the simple equation

$$Ra = (l^2\alpha^2 + \pi^2)^2/l^2\alpha^2, \tag{1}$$

where $l = 1, 2, \ldots$ gives the number of rolls within the box. For $Ra$ greater than the minimum critical Rayleigh number $4\pi^2$, (1) determines a range of $\alpha$, depending on $l$, for which two-dimensional roll convection is *a priori* possible. Consider $Ra = 100$, for example. Single alpha ($s\alpha$) rolls ($l = 1$) are possible for $0{\cdot}35 \lesssim \alpha/\pi \lesssim 2{\cdot}83$, while double alpha ($d\alpha$) rolls ($l = 2$) can exist for $0{\cdot}18 \lesssim \alpha/\pi \lesssim 1{\cdot}41$. The vertical lines in figure 1 divide this $\alpha, \beta$ diagram (for $Ra = 100$) into regions in which $s\alpha$ and $d\alpha$ convective rolls can exist, according to linear theory. The arrows adjacent to the lines point in the directions of allowable box sizes. The analysis for $\beta$ rolls (two-dimensional rolls aligned with the $x$ axis) is identical, and the horizontal lines in figure 1 delineate values of $\beta$ for which $s\beta$ and $d\beta$ convective rolls are possible.

Given the ranges of box sizes for which various modes of multicellular two-dimensional convection can *a priori* occur, we next determine, numerically, the actual form of the finite amplitude, steady, two-dimensional convection for given values of $Ra$ and $\alpha$ (or $\beta$). A Galerkin procedure is employed, based on a truncated Fourier decomposition of the flow and temperature fields. An analysis of the stability of the convection to roll disturbances of infinitesimal amplitude perpendicular to the basic roll pattern is then carried out. When the finite amplitude convection is found to be unstable, only the disturbance with the maximum growth rate is considered. This mode is always found to be non-oscillatory; there may be oscillatory modes with smaller growth rates. The results of this stability analysis for $Ra = 100$ are given in figure 1.

Consider first the stability of $s\alpha$ rolls to $s\beta$ disturbances. Figure 1 shows two curves labelled $s\alpha us\beta$ (single $\alpha$ unstable to single $\beta$) which, together with the relevant horizontal and vertical lines, define regions of the $\alpha$, $\beta$ plane in which $s\alpha$ rolls are unstable to $s\beta$ disturbances. The arrows point towards the regions of instability. From the figure it can be seen that $s\alpha$ rolls are stable against all $s\beta$ disturbances for $0{\cdot}72 \lesssim \alpha/\pi \lesssim 1{\cdot}77$. For other values of $\alpha$, there are certain ranges of $\beta$ for which the $s\alpha$ rolls are unstable. For example, for $\alpha/\pi = 0{\cdot}6$, $s\alpha$ rolls are unstable to $s\beta$ disturbances when $\beta/\pi$ is

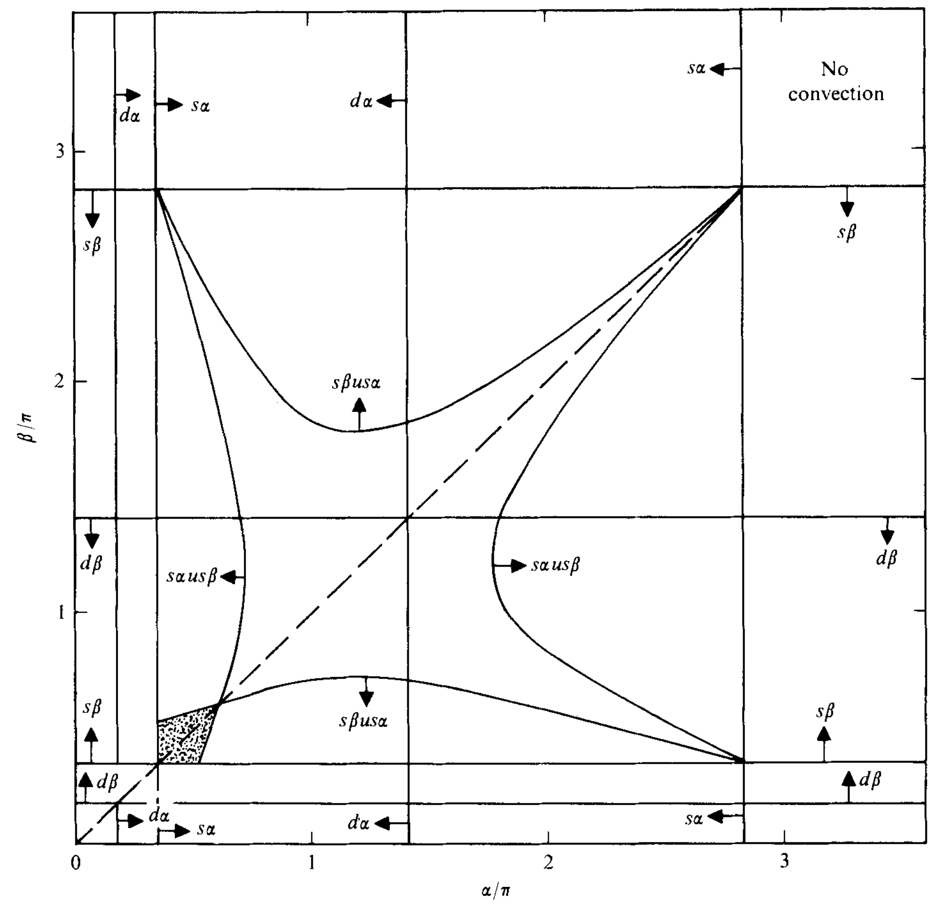


FIGURE 1. Stability of steady two-dimensional convective rolls in a box with horizontal dimensions $\pi d/\alpha$ and $\pi d/\beta$ for $Ra = 100$. Horizontal and vertical lines denote ranges of box sizes for which single alpha ($s\alpha$), double alpha ($d\alpha$), single beta ($s\beta$) and double beta ($d\beta$) rolls can exist on the basis of linear stability theory. The curves denoted $s\alpha us\beta$ delineate box dimensions for which finite amplitude $s\alpha$ rolls are unstable to infinitesimal $s\beta$ rolls; the arrows point to regions of parameter space in which $s\alpha$ rolls are unstable. The curves denoted $s\beta us\alpha$ have a similar interpretation. In the shaded region, $s\alpha$ and $s\beta$ rolls are unstable to each other.

between about 0·55 and 1·93; other $\beta/\pi$ ranges for which $s\alpha$ rolls are unstable are 0·35 to 2·67 for $\alpha/\pi = 0{\cdot}4$ and 0·65 to 2·17 for $\alpha/\pi = 2{\cdot}3$. Mirror images of the $s\alpha us\beta$ curves about the diagonal $\alpha = \beta$ give the $s\beta us\alpha$ curves delineating, in part, the regions of the $\alpha$, $\beta$ plot in which $s\beta$ rolls are unstable to $s\alpha$ disturbances.

The shaded region of figure 1 is of particular interest; within it $s\alpha$ rolls are unstable to $s\beta$ disturbances, but $s\beta$ rolls of finite amplitude are also unstable to $s\alpha$ disturbances. Thus if only single cells were possible ($l = 1$), convection in boxes whose dimensions lay in the shaded area of figure 1 would be either oscillatory or steady and three-dimensional.

The situation is more complicated than figure 1 shows because of the possibility of multicellular two-dimensional convection. Thus the shaded area of the figure cannot be identified as a region of three-dimensional steady convection until we have considered the stability of steady double-cell rolls. The figure shows that, on the basis of linear theory, double rolls can occur for parameter values lying in the shaded area. If we were to ascertain that in the shaded area (or in a portion of it) double $\alpha$ or $\beta$ rolls were unstable, then the stability of triple-cell rolls would have to be investigated. Indeed, before it could be concluded that in a region of the $\alpha$, $\beta$ plane steady convection would necessarily be three-dimensional, the stability of all multicellular convective roll patterns which could exist in the region, according to linear theory, would have to be considered. It is not necessary to go beyond the double-cell roll, to even larger numbers of rolls, to preclude the necessity of any three-dimensional steady convection in the shaded area of figure 1 because, as we shall see shortly, steady double-cell convective rolls are stable forms of convection therein.

By appropriately scaling the curves in figure 1, we can easily delineate the regions of $\alpha$, $\beta$ space which correspond to stability of multicellular convective rolls to multicellular roll disturbances normal to the basic pattern. Consider, for example, the stability of $s\alpha$ rolls to $d\beta$ disturbances (double-cell $\beta$ roll disturbances). For a given $\alpha$, the $s\alpha us\beta$ curves in figure 1 determine a range of $\beta$ for which the $s\alpha$ convection is unstable to single $\beta$ rolls. The physical situation is identical for a box whose dimension in the $y$ direction is twice as large. Thus $s\alpha$ rolls in such a larger box would be unstable to double-cell $\beta$ roll disturbances; the $\beta$ values would be half of those which lead to instability via single $\beta$ disturbances in the smaller box. To generate the $s\alpha ud\beta$ curves in figure 2, we need only to halve the ordinates of the $s\alpha us\beta$ curves in figure 1 at fixed $\alpha$. The $s\beta ud\alpha$ curves in figure 2 are mirror images about the line $\alpha = \beta$ of the $s\alpha ud\beta$ curves. It is now clear how to generate all the curves in figure 2: $d\alpha us\beta$ is obtained by halving the abscissae of the $s\alpha us\beta$ curves for fixed $\beta$, $d\alpha ud\beta$ is obtained by halving the ordinates of the $d\alpha us\beta$ curves at fixed $\alpha$, and so forth.

Figure 2 completely delineates the stability of single- and double-cell convective rolls in rectangular boxes of arbitrary size to orthogonal single- and double-cell roll disturbances for $Ra = 100$. It can be seen at once from figure 2 that the region which was shaded in figure 1 and identified as a possible area of steady three-dimensional convection lies almost entirely within a region of $\alpha$, $\beta$ space in which double-cell $\alpha$ or $\beta$ rolls are each stable forms of two-dimensional convection. Only in very narrow slivers on the borders of the shaded quadrilateral in figure 1 is it possible to decide, on the basis of the stability curves in figure 2, whether $d\alpha$ or $d\beta$ rolls are preferred. Within the shaded region of figure 2, $s\alpha$ or $s\beta$ convection rolls cannot exist, while $d\alpha$ and $d\beta$ rolls are unstable. Thus this area is a second potential region of $\alpha$, $\beta$ space in which steady convection might have to be three-dimensional. However, triple alpha ($t\alpha$) and triple beta ($t\beta$) rolls could exist in the boxes of this second shaded region, and the stability of these and even higher-order modes would need to be considered before concluding anything about the necessity of three-dimensional convection.

We have carried the analysis up to and including $t\alpha$ and $t\beta$ modes for $Ra = 100$ without being able to identify a region of $\alpha$, $\beta$ space in which steady convection is necessarily three-dimensional; some two-dimensional, perhaps multicellular roll pattern is always a stable configuration. The existence of three-dimensional steady convection at this Rayleigh number would depend on a non-uniqueness associated with initial conditions.

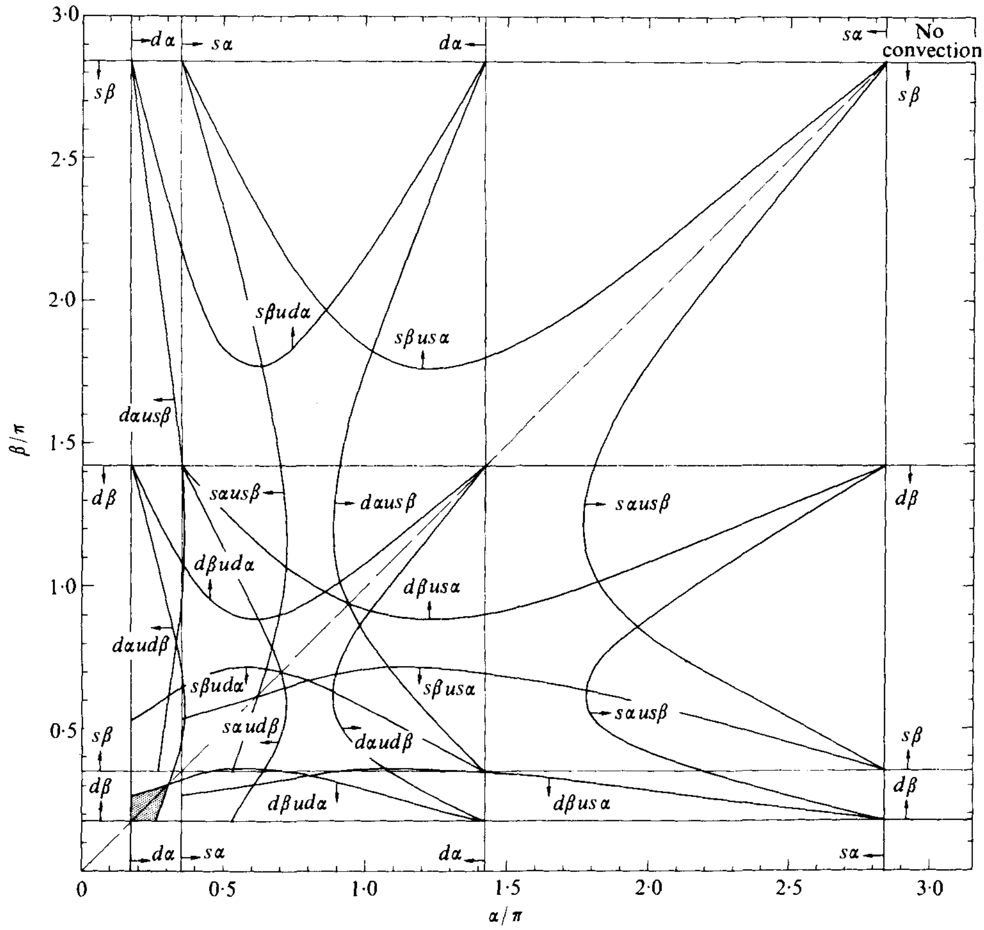


FIGURE 2. Stability of single- and double-cell convective rolls for $Ra = 100$. In the shaded region, $s\alpha$ and $s\beta$ rolls cannot exist and $d\alpha$ and $d\beta$ rolls are unstable.

## 3. Results for $Ra = 200$, 340 and 400

The procedure we have described in some detail for $Ra = 100$ has also been carried out for other values of $Ra$; the results are shown in figures 3–5 for $Ra = 200$, 340 and 400, respectively. As was the case for $Ra = 100$, it is not possible to delineate a range of box sizes for which steady convection would necessarily be three-dimensional when $Ra = 200$. This conclusion is based only on the stability curves shown in figure 3, and we cannot rule out the possibility that an extension of these curves to the higher-order multicellular modes might define regions of $\alpha$, $\beta$ parameter space which require steady convection to be three-dimensional.

By comparing figures 2 and 3 one can see significant distortions of the stability curves which eventually produce the shaded regions of $\alpha$, $\beta$ space in figures 4 and 5, corresponding to boxes in which steady convection is necessarily three-dimensional at the higher values of Rayleigh number appropriate to the latter figures. Thus, for Rayleigh numbers larger than some value between 200 and 340, it becomes possible, with our approach, to delineate box dimensions requiring steady convection to be

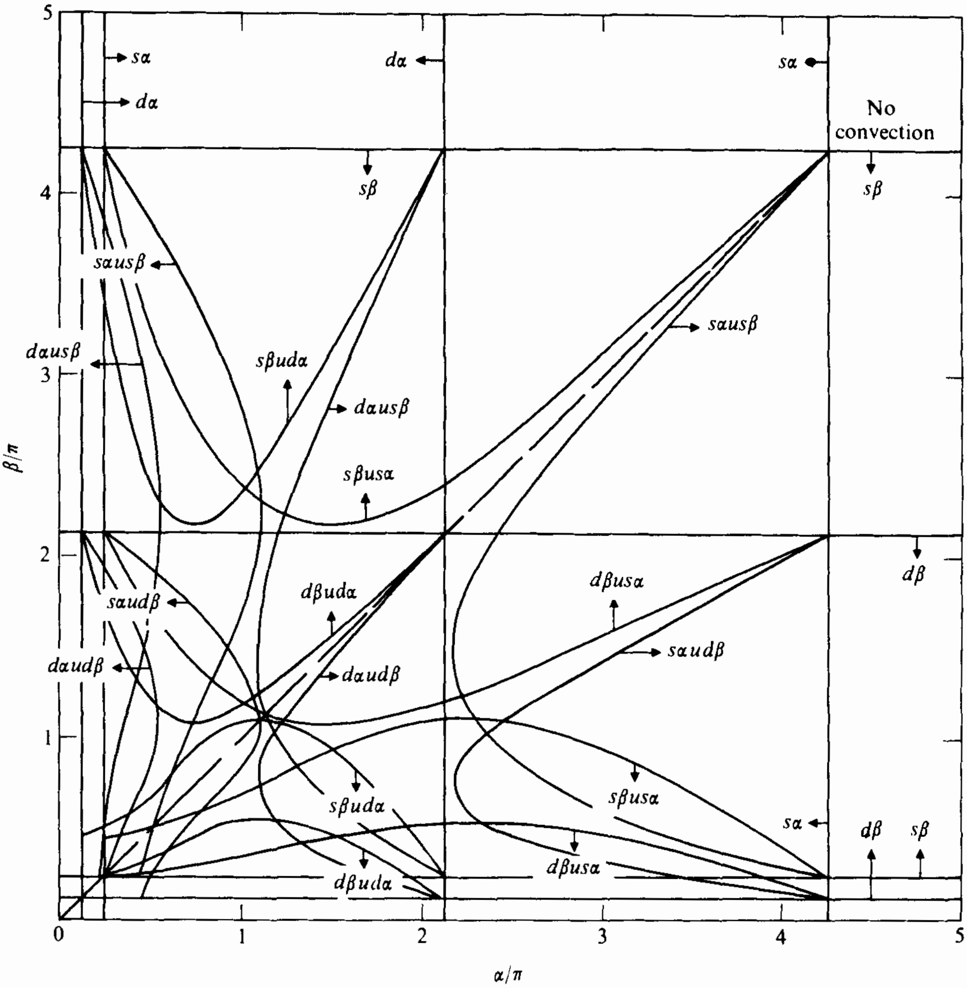


FIGURE 3. Stability curves for $Ra = 200$.

three-dimensional. The additional constraints on the flow imposed by the side walls of the box can be appreciated by recalling that between infinite parallel planes steady convection is necessarily three-dimensional only for $Ra \gtrsim 380$.

At $Ra = 340$, there are many regions of parameter space in which steady convection must be three-dimensional. For the purposes of discussion, we focus on one of these regions: the shaded region bounded by the curves *sβusα*, *sαusβ* and *dβusα* in figure 4. A similar region does not exist at $Ra = 100$ (figure 2) since the *sβusα* curve does not intersect either the *sαusβ* or the *dβusα* curve. At $Ra = 200$ (figure 3), a similar region is also absent, but the *sβusα* curve is now seen to intersect the *sαusβ* curve and to lie much closer to the *dβusα* curve than it did at $Ra = 100$ (figure 2). The development with the Rayleigh number of this unstable region in figure 4 can thus be traced through figures 2 and 3. On the basis of figures 2–4, one can estimate that a similar unstable region should first occur for some value of $Ra$ much closer to 200 than to 340.

That in this particular region of figure 4 steady convection must indeed be three-dimensional can be argued as follows. First, the region lies completely in that portion

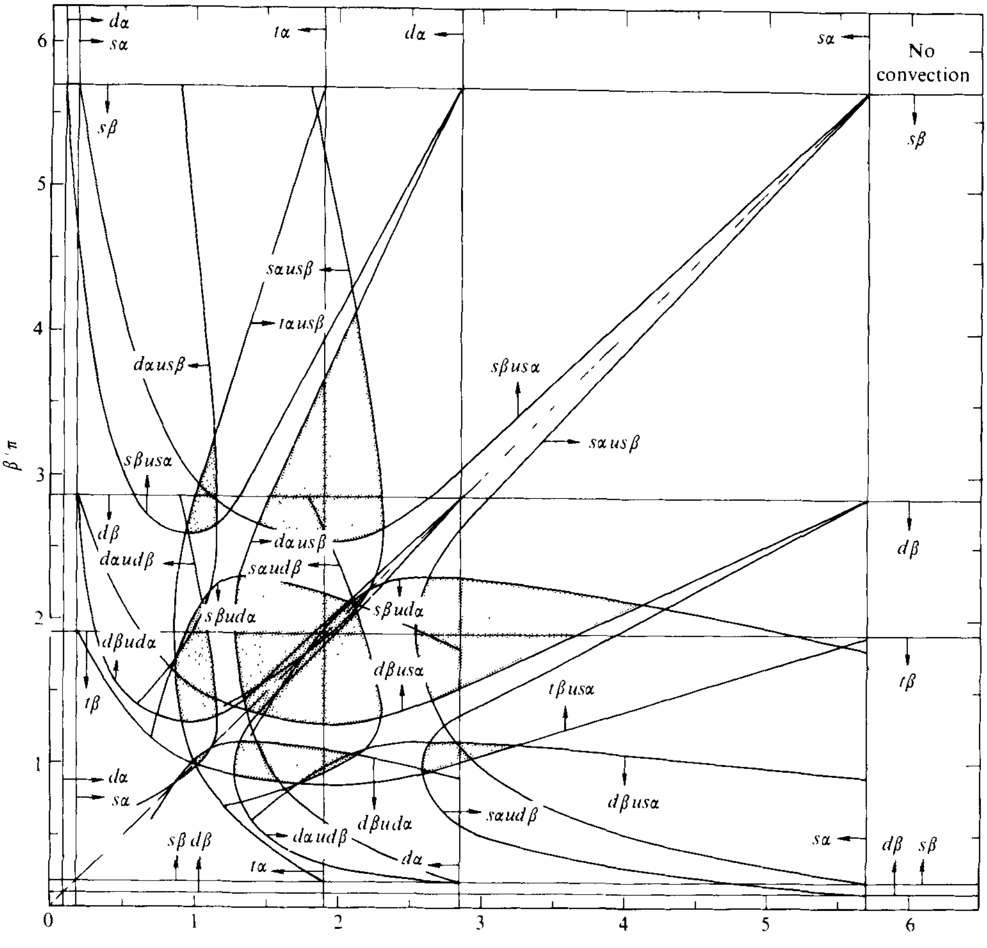


FIGURE 4. Stability curves for $Ra = 340$. Within the shaded regions, convection must be either unsteady or steady and three-dimensional. Note that $t\alpha us\beta$ and $t\beta us\alpha$ curves are required to delineate some of the shaded areas.

of $\alpha$, $\beta$ space in which $s\alpha$ or $s\beta$ steady convection is *a priori* possible. However, from the stability curves, it is clear that neither $s\alpha$ or $s\beta$ could exist since each is unstable to the other. Within a portion of this region $d\alpha$ convection is *a priori* possible. However, this area lies far to the right of the $d\alpha us\beta$ curve, where $d\alpha$ convection is unstable to $s\beta$ rolls. Thus steady $d\alpha$ convection cannot occur for box sizes in the shaded region. Neither could any higher-order multicellular $\alpha$ rolls exist in this shaded region of figure 4 since, for example, $t\alpha$ convection could not exist for $\alpha/\pi \gtrsim 1{\cdot}9$. Within this region of figure 4, $d\beta$ and $t\beta$ cells are also *a priori* possible configurations for steady convection. But the positions of the stability curves $d\beta us\alpha$ and $t\beta us\alpha$ ensure that neither of these modes could in fact exist. Thus all possible forms of steady two-dimensional convection in this region have been shown to be unstable. One must conclude that either no steady convection is possible for such box sizes at $Ra = 340$ or steady convection is fully three-dimensional. By analogous reasoning, the reader may verify that in the remaining shaded regions of figure 4 steady convection would be three-dimensional.

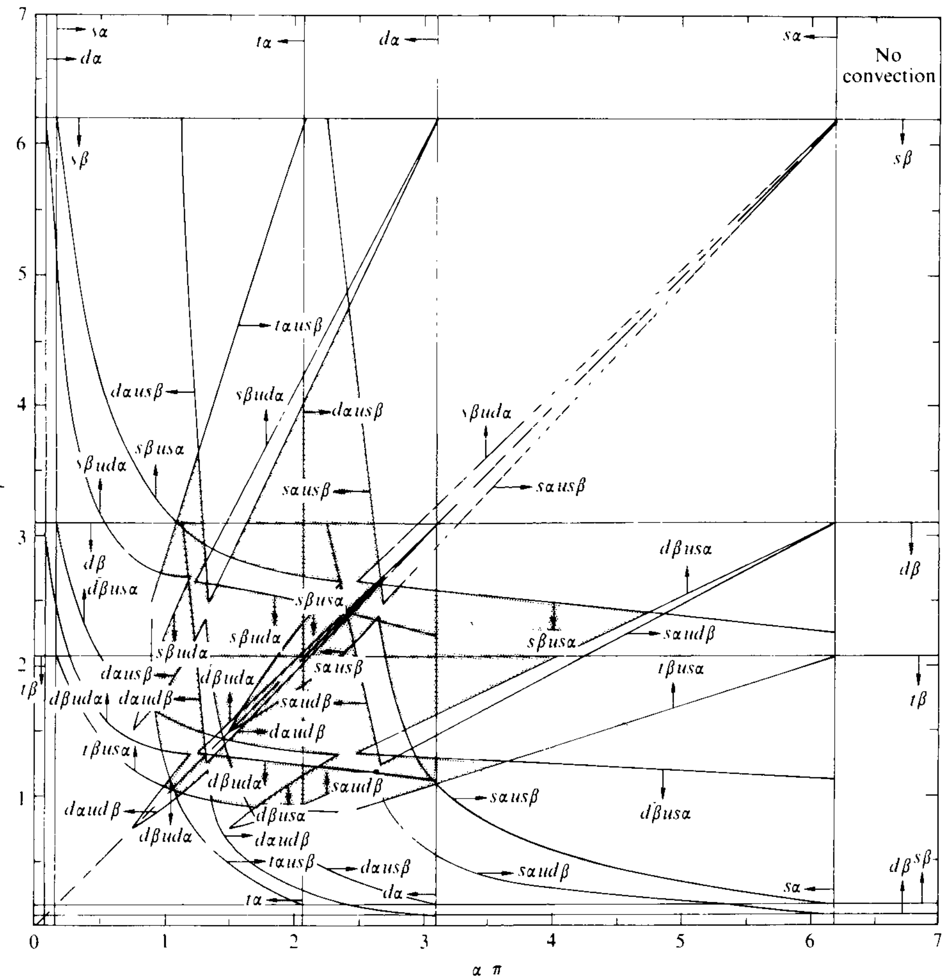


FIGURE 5. Same as figure 4 for $Ra = 400$.

The shaded regions of parameter space in figure 5 define those box sizes for which steady convection is necessarily three-dimensional at $Ra = 400$. The regions are similar to the ones shown in figure 4 for the lower Rayleigh number. As $Ra$ increased from 340 to 400, the regions of parameter space in which steady convection would have to be three-dimensional increased in size and coalesced into the simply connected shaded domain of figure 5.

## 4. Summary and discussion

We have outlined an approach whereby one can predict the dimensions of a box in which convection would necessarily be either unsteady or steady and fully three-dimensional. Thus investigators interested in three-dimensional convection will be able to choose appropriate box sizes for their studies. The approach has been applied to thermal convection in a box containing saturated porous material. For $Ra = 100$ and 200, it appears unlikely that there are any box dimensions for which there is not a stable (possibly multicellular) two-dimensional steady motion. Box dimensions have been found for which convection must either be unsteady or steady and three-dimensional at Rayleigh numbers of 340 and 400. Steady two-dimensional convection between infinite parallel planes is necessarily unstable only for $Ra \gtrsim 380$, but when the rolls are forced to fit within a box, they are unstable, for at least some box sizes, at substantially lower Rayleigh numbers. For $Ra = 340$ and 400, we have shown that two-dimensional rolls are unstable in boxes which have heights ranging from one to four or five times their horizontal dimensions. Because of the complexity introduced by the possibility of high-order multicellular convection at these Rayleigh numbers, we have not attempted to investigate the situation for boxes which are both wider and longer than they are tall.

Our approach could also be applied to thermal convection in an ordinary viscous fluid in a box by extending the work of Busse (1967) to determine the range of cross-roll wavenumbers which destabilize a basic finite amplitude convective roll of given wavenumber and Rayleigh number between infinite parallel planes. This could be done only if the side walls of the box were slippery boundaries. If the sides of the box were no-slip surfaces, then a completely new study of the stability of convective rolls in a viscous fluid in a box would be required. Porous-medium convection has the advantage, for our method, of allowing slip parallel to confining rigid boundaries.

We stress again that steady three-dimensional convection might occur at Rayleigh numbers and box sizes for which two-dimensional convective rolls are stable forms of convection if the form of convection has a non-uniqueness associated with initial conditions. Finally, although we have not emphasized this, the stability curves presented here can be used to determine those multicellular rolls which are stable for a given Rayleigh number and box size.

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